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LETTER TO THE EDITOR

Order propagation in dilute antiferromagnetic Potts models

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Abstract. The possible orderings in a dilute antiferromagnetic $q=3$ Potts model on a triangular lattice are studied by applying to a hierarchical lattice an approximate renormalisation group scheme which incorporates and preserves all the symmetries of the Potts model. This scheme indicates that states with *partial* order, in which only a subset of the symmetries are broken, can exist in systems with these symmetries. They do not, however, occur in the model with nearest-neighbour pair interactions only. In this model we find solely the antiferromagnetically ordered states which occur for bond probabilities in excess of p_1, which is greater than p_c, the critical probability for bond percolation.

Conventional wisdom for quenched dilute spin systems is that the zero-temperature transition occurs at p_c, the critical probability for site (or bond) percolation. It has, however, become apparent that some randomly diluted systems do not undergo transitions to an ordered phase until p exceeds p_c by some non-zero amount. This is the case for elastic networks (Feng and Sen 1984, Wang and Harris 1985, Tremblay *et al* 1986), for the quenched bond (Ono 1983) and site (Adler *et al* 1986) dilute $q=3$ antiferromagnetic Potts model on the triangular lattice, and for systems with frustrated ground states (Shnidman and Mukamel 1984) such as quadrupoles on FCC and triangular lattices (Adler *et al* 1986). The latter are models of mixtures of ortho- and para-hydrogen or deuterium which exhibit long-range orientational order of the quadrupole moments only for concentrations greater than some value x_c which is well in excess of p_c (Banke *et al* 1985, Kubik *et al* 1985, Harris and Meyer 1985). Bounds for x_c were obtained by Adler *et al* (1986) who also suggested that between x_c and p_c a partial ordering was possible. However, neither the nature nor the existence of this phase was shown by them.

In this letter we study the properties of a model system on a hierarchical lattice which is designed to mimic the properties of a $q=3$ antiferromagnetic Potts model on the triangular lattice with quenched bond dilution. Within an approximate renormalisation group calculation, we find that the zero-temperature transition to one of the six equivalent ground states with antiferromagnetic order occurs only for bond concentrations $p \geqslant p_1$ where $p_1 > p_c$. Thus, the fact that the lattice is geometrically percolated is insufficient to guarantee the existence of the order. Of greater interest is our result that systems possessing the symmetry of the $q=3$ antiferromagnetic Potts model *can* exhibit states of partial order, e.g. ones exhibiting no long-range antiferromagnetic order but still a long-range helicity order. Such states have, thus far, been neither

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considered nor sought. Although they do not occur in the model with nearest-neighbour interactions only, we believe that they can be brought about by the dilution of models with additional interactions.

With all sites (and bonds) occupied, the $q=3$ antiferromagnetic Potts model on the triangular lattice has six degenerate ground states of the form shown in figure 1 (Schick and Griffiths 1977). The states of the Potts spins are labelled A, B and C and the three sublattices of the tripartite lattice by α, β and γ. Note that a helicity, or handedness, can be assigned to each of the six ground states according to the order in which the spin states a, b and c are encountered on, say, the upward-pointing triangles (Lee *et al* 1984). Ground states 1, 2 and 3 have one helicity and states 4, 5 and 6 have the opposite. Upon quenched dilution of either occupied sites or bonds, finite clusters which are sufficiently connected will continue to maintain the order of one particular ground state. However, in those localities where many sites no longer have three occupied nearest neighbours, the antiferromagnetic order in one region will fail to be propagated to another, even though the sample may be well above the geometrical percolation threshold. As shown by Adler *et al* (1986), those connections which fail to pass the antiferromagnetic order *do* pass some information; for example,

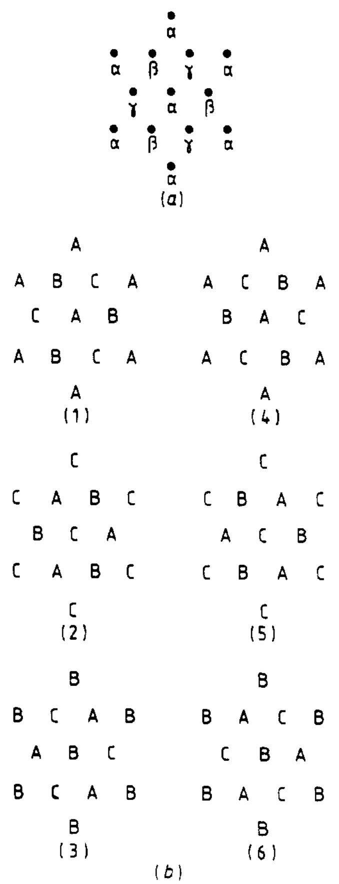

Figure 1. (*a*) Section of the triangular lattice showing the three sublattices α, β and γ. (*b*) The same section is shown occupied by spins in states A, B or C in the six possible antiferromagnetic ground states.

even a single isolated bond from one region to a given site eliminates one of the three possible spin values from occurring at that site and hence reduces from six to four the number of ground states which that site can be a part of. This reduction is information. Together, such links connecting one region to another can pass the antiferromagnetic order. Therefore, the determination of the concentration below which such order cannot be passed requires a knowledge of the global lattice connectivity, a circumstance which makes the determination formidable.

To circumvent this problem, we consider a model on a hierarchical lattice which is designed to mimic the original Potts problem on the triangular lattice. Our choice is shown schematically in figure 2. There are regions, or vertices, which can be in one of six states. A vertex on one level of the hierarchy is connected by six bonds to a vertex on the next level. Each bond is labelled by a vector, $\boldsymbol{v}_i$, whose components v_{ij} give the conditional probability that the second vertex would be found in state j given that the first vertex were in state i and the two vertices were connected by only this bond. It is these probabilities which will undergo renormalisation and it is in the specification of their initial values that the original problem, the antiferromagnetic Potts model on the triangular lattice with nearest-neighbour pair interactions only, is simulated. For example, as noted above, a single bond which connects a region ordered in ground state 1 with another site transmits the information that the site can be part of a region ordered in any of four possible ground states, state 1 and three others with equal probability. Which three others depends on which sublattices the bond connects. Upon averaging over the six possibilities, one finds that the probability that the site can be part of a region ordered in ground state j is given by the jth component of the vector $\boldsymbol{v}_1^0 = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. This, therefore, is the representation in the hierarchical model of an average bond in the original Potts problem. Similarly, if a particular bond in the Potts problem between an ordered region in state 1 and another site were missing, then this site could be part of a region ordered into any one of the six possible ground states with equal probability. Therefore, this is represented in the hierarchical model by a bond $\boldsymbol{v}_1^{\mathrm{d}} = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. It follows that the initial condition which represents the quenched dilute antiferromagnetic Potts model on the triangular lattice is $\boldsymbol{v}_1^{\mathrm{i}} = p\boldsymbol{v}_1^0 + (1-p)\boldsymbol{v}_1^{\mathrm{d}}$. The other v_{ij} for $i \neq 1$ can be found from this as they are not independent, being related by the symmetry operations of the original Potts problem (Schick and

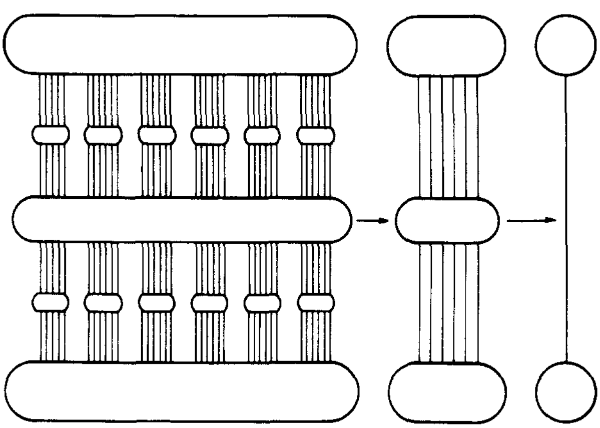

Figure 2. Hierarchical lattice. In going from one level of the lattice to another, six bonds are replaced by (or replace) one bond.

Griffiths 1977). These operations reduce from 36 to 6 the number of independent elements v_{ij} of the matrix $\boldsymbol{v}$, which has the form

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{y}^{\mathrm{T}} & \boldsymbol{x}^{\mathrm{T}} \end{pmatrix}$$

where

$$\boldsymbol{x} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{13} & v_{11} & v_{12} \\ v_{12} & v_{13} & v_{11} \end{pmatrix}$$

and

$$\boldsymbol{y} = \begin{pmatrix} v_{14} & v_{15} & v_{16} \\ v_{16} & v_{14} & v_{15} \\ v_{15} & v_{16} & v_{14} \end{pmatrix}.$$

The normalisation condition further reduces this number to 5.

The recursion relation that represents the transformation from one level of the hierarchy to the next is illustrated in figure 3. There are two steps in the renormalisation process. In the first, the six bonds v_{ik} which link two regions ordered in states i and k are replaced by a single effective bond P_{ik}. We treat the probabilities associated with each bond as independent so that the components of the normalised effective bond are given by

$$P_{ik} = \frac{v_{ik}^6}{(\Sigma_l v_{il}^6)^{-1}}. \tag{1}$$

This is also an approximation. In the second step, we sum out the intermediate region to obtain the renormalised components v'_{ij}:

$$v'_{ij} = \sum_k P_{i,k} P_{k,j}. \tag{2}$$

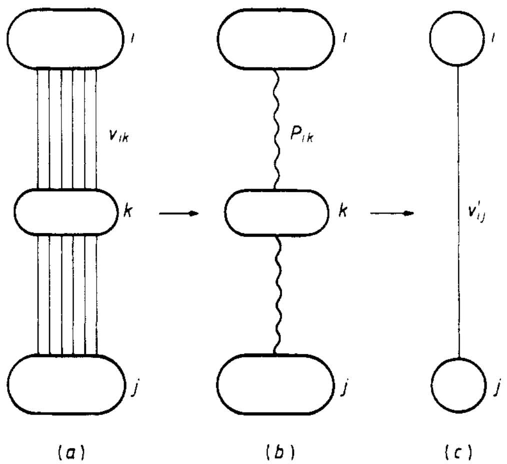

Figure 3. Two-stage renormalisation transformation, shown schematically. It is given explicitly by (1) and (2).

Equations (1) and (2) constitute the renormalisation scheme. It is an approximate scheme because at each level of the transformation we consider an average bond and not a distribution of bonds.

We have found six fixed-point vectors which represent sinks of the flow. The sink $(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$ represents the disordered phase. The sink $(1,0,0,0,0,0)$ represents the antiferromagnetically ordered phase because such a bond propagates, with probability unity, the order of the region from which it comes. The other four sinks represent different possible states of *partial* order, i.e. states in which not all of the symmetries of the original system are broken. The first of these $(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,0)$ represents a state in which only the helicity symmetry is broken; thus, a site connected to a region ordered in state 1 by this bond could belong, with equal probability, to a region ordered in any of the three states which have the same helicity as state 1. The three sinks $(\frac{1}{2},0,0,\frac{1}{2},0,0)$, $(\frac{1}{2},0,0,0,\frac{1}{2},0)$, and $(\frac{1}{2},0,0,0,0,\frac{1}{2})$ represent states in which only one of the sublattices is ordered. There is also a 2-cycle between $(0,1,0,0,0,0)$ and $(0,0,1,0,0,0)$. As this represents an alternation between two different states as the length scale increases (McKay *et al* 1982), this probably is an artefact of the hierarchical lattice, although we cannot exclude the possibility in more realistic systems of phases which correspond to cyclic or chaotic trajectories in the parameter space. We have also located nine unstable fixed points which govern separatrixes between some of the above sinks. They are given in table 1.

Table 1. Fixed-point vectors (other than sinks). They are of the form $p^*v_1^A+(1-p^*)v_1^B$ with v_1^A, v_1^B and p^* given in the table along with the number of relevant eigenvalues. The fourth and fifth entries have, by symmetry, two partners which are not shown explicitly.

v_1^A	v_1^B	p^*	Number of relevant eigenvalues
$(1,0,0,0,0,0)$	$(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$	0.0197	5
$(1,0,0,0,0,0)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,0)$	0.0252	2
$(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,0)$	$(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$	0.0283	1
$(1,0,0,0,0,0)$	$(\frac{1}{2},0,0,\frac{1}{2},0,0)$	0.0283	1
$(\frac{1}{2},0,0,\frac{1}{2},0,0)$	$(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$	0.0252	1

The phase diagram of the original Potts model is obtained by scanning along the physical line $v_{1j}(p)=pv_{1j}^0+(1-p)v_{1j}^d$. We find that the system is disordered for all $p\leq p_1=0.1828$, and is antiferromagnetically ordered otherwise. This is to be compared with p_c, the bond percolation threshold of our hierarchical lattice which can be determined as follows. Suppose that on this lattice we employ, with probability p, the bonds $v_{ij}^s=(1,0,0,0,0,0)$ which pass order and, with probability $(1-p)$, the bonds $v_{ij}^d=(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$ which do not. The probability at which the former bonds percolate is p_c. Referring to the left-hand part of figure 2, we see that the probability that there are no such bonds between the upper region and the middle region is $(1-p)^6$, so that the probability that there are no such bonds between the upper and lower regions is $2(1-p)^6-(1-p)^{12}$, where the second term compensates for a double counting. Thus, p_c is determined exactly from $(1-p_c)=2(1-p_c)^6-(1-p_c)^{12}$, which yields $p_c=0.0327$. Our approximate recursion relation yields an approximation to p_c which is found by scanning the line $v_{1j}=pv_{1j}^s+(1-p)v_{1j}^d$ for the separatrix between ordered and disordered phases. This yields $p_c'=0.0197$ and is the value with which the threshold $p_1=0.1828$, derived from the recursion relation, should be compared.

Summarising, we have presented an approximate renormalisation group on a hierarchical lattice which yields the result that, upon dilution, the antiferromagnetic order is destroyed at a dilution p_1 which is greater than p_c. Between p_c and p_1 the lattice, while geometrically percolated, cannot sustain antiferromagnetic order. We have also found that states which exhibit *partial* order, such as long-range helicity order, but no long-range antiferromagnetic order should occur in some models governed by Hamiltonians with the same symmetries as that of the $q=3$ Potts antiferromagnet but with additional interactions. Such models are currently under investigation.

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